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On Irregular Coloring of Some Generalised Graphs

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Abstract

Irregular coloring was introduced by Radcliffe and Zhang in 2006. Irregular coloring follows the condition: (i) proper coloring, (ii) distinct vertices have distinct color codes. The irregular chromatic number denoted by χ_{ir}. In this paper, we find the irregular chromatic number for the graphs, $M(nW_m)$, $T(nW_m)$, $L(nW_m)$, $C(nW_m)$, $M(nF_m)$, $T(nF_m)$, $L(nF_m)$, $C(nF_m)$, $S(G)$ and $\mu(G)$.

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1 Introduction

In this paper we consider only simple, undirected and connected graphs. For a positive integer k and a proper coloring $c : V(G) \to \{1, 2, \ldots, k\}$ of the vertices of a graph G,

the *color code* [9] of a vertex v of G (with respect to c) is the ordered $(k+1)$-tuple $code_c(v) = (a_0, a_1, \dots a_k)$ where a_0 is the color assigned to v (that is $c(v) = a_0$) and for $1 \leq i \leq k$, a_i is the number of vertices adjacent to v that are colored i.
The coloring c is called *irregular*[8] if distinct vertices of G have distinct color codes. The *irregular chromatic number* of G is denoted by $\chi_{ir}(G)$.
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *middle graph* [2] of G, denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

(i) x, y are in $E(G)$ and x, y are adjacent in G.

(ii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

The *total graph* [14] of G, denoted by $T(G)$ is defined as follows: The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

(i) x, y are in $V(G)$ and x is adjacent to y in G.

(ii) x, y are in $E(G)$, x, y are adjacent in G.

(iii) x is in $V(G)$ and y is in $E(G)$, and x, y are incident in G.

The *central graph* [14] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The *line graph* [4] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

Radcliffe and Zhang were introduced the concept irregular coloring in [8] and discussed the irregular chromatic bounds for the disconnected graph in [9]. The following papers [1, 6, 10, 11] are given some more results on irregular colorings. The neighbourhood of a vertex u in a graph G is $N(u) = \{v \in V(G) : uv \in E(G)\}$. In this paper, we have considered the following useful result from [7].

Let c be a (proper) coloring of the vertices of a nontrivial graph G and let u and v be two vertices of G then

$$If \text{c(u)} \neq c(v), \text{ then } code(u) \neq code(v). \tag{1}$$

$$If \text{d(u)} \neq d(v), \text{ then } code(u) \neq code(v). \tag{2}$$

$$If \text{c } is\ irregular\ and \text{ N(u)=N(v)}, \ then \text{ c(u)} \neq c(v). \tag{3}$$

A *wheel graph* W_n [12] is defined as the graph $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph on n vertices.

An **n-wheel graph** is a graph that can be constructed by $nC_m + K_1$, for $m \geq 3$, $n \geq 1$. It consists of n copies of cycle of order m, where the vertices of all cycles are joined to a common hub and is denoted by nW_m with size $2mn$ and order $nm + 1$.

An **n-fan graph** is a graph that can be constructed by $nP_m + K_1$, for $m \geq 3$, $n \geq 1$. It consists of n copies of path of order m, where the vertices of all paths are joined to a common hub and is denoted by nF_m.

For each point v of a graph G, take a new point $v^{'}$. Join $v^{'}$ to all points of G adjacent to v. The graph $S(G)$ thus obtained is called the *Splitting graph* [3, 13] of G.

Consider a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Apply the following steps to thegraph G

1. Take the set of new vertices $U = \{u_1, u_2, \ldots, u_n\}$ and add edges from each vertex u_i of U to the vertices v_j if the corresponding vertex v_i is adjacent to v_j in G,

2. Take another new vertex u and add edges to all elements in U.

The new graph thus obtained is called the *mycielski graph or mycielskian* of G and is denoted by $\mu(G)$ [3, 5].

2 Irregular Coloring of middle graph, total graph, central graph and line graph of n-wheel Graph

Theorem 2.1. *For $m \geq 4$ and $n \geq 1$, the irregular chromatic number of n-wheel graph is $\chi_{ir}(M(nW_m)) = mn+1$.*

Proof. The vertex set $V\left[M\left(nW_m\right)\right] = \Big\{x, a_i^j, e_i^j, f_i^j, g_{1m}^j :$ $1 \leq j \leq n, 1 \leq i \leq m\}$, where e_i^j are the vertices on the edges xa_i^j for $1 \leq j \leq n, 1 \leq i \leq m$, f_i^j are the vertices on the rim edges $a_i^j a_{i+1}^j$ of nW_m for $1 \leq j \leq n, 1 \leq i \leq m-1$ and g_{1m}^j are the vertices on the rim edges $a_1^j a_m^j$ of nW_m for $1 \leq j \leq n$.

Consider the $(mn+1)$-coloring for the given $M(nW_m)$ as follows:

- $x = 1$
- $e^1_k = k+1$ for $1 \leq k \leq m$
- $e^2_{k-m} = k+1$ for $m+1 \leq k \leq 2m$
- $e^3_{k-2m} = k+1$ for $2m+1 \leq k \leq 3m$
  $\vdots$
- $e^n_{k-(n-1)m} = k+1$ for $(n-1)\,m+1 \leq k \leq nm$
- For $1 \leq i \leq m$, $1 \leq j \leq n$ $a^j_i = 1$
- $g^k_{1m} = k + n(m-1)$ for $1 \leq k \leq n$
- $f^1_k = k$, $2 \leq k \leq m-1$
- $f^2_{k-(m-1)} = k$ for $m \leq k \leq 2(m-1)$
- $f^3_{k-2(m-1)} = k$ for $2m-1 \leq k \leq 3(m-1)$
  $\vdots$
- $f^n_{k-(n-1)(m-1)} = k$ for $(m-1)\,n-m+2 \leq k \leq n(m-1)$
- $f^1_1 = mn+1.$

All $d\left(f^j_i\right) \neq d\left(e^j_i\right)$ which implies that *code* $\left(f^j_i\right) \neq$ *code* $\left(e^j_i\right)$. Hence, $\chi_{ir}(M(nW_m)) \leq mn+1$. The set $\left\{x, e^j_i : 1 \leq j \leq n,\right.$ $\left.1 \leq i \leq m\right\}$ produce a clique of order $mn+1$. Therefore, $\chi_{ir}(M(nW_m)) \geq mn+1$. Thus, $\chi_{ir}(M(nW_m)) = mn+1$.

□

Theorem 2.2. *For $m \geq 4$ and $n \geq 1$, the irregular chromatic number of n-wheel graph is $\chi_{ir}(T(nW_m)) = mn + 1$.*

Proof. The vertex set $V[T(nW_m)] = \Big\{x, a_i^j, e_i^j, f_i^j, g_{1m}^j :$ $(1 \leq j \leq n), (1 \leq i \leq m)\}$, where e_i^j are the vertices on the edges xa_i^j for $(1 \leq j \leq n), (1 \leq i \leq m)$, f_i^j are the vertices on the rim edges $a_i^j a_{i+1}^j$ of nW_m for $(1 \leq j \leq n)$,
$(1 \leq i \leq m-1)$ and g_{1m}^j are the vertices on the rim edges $a_1^j a_m^j$ of nW_m for $(1 \leq j \leq n)$.
Consider the $(mn+1)$-coloring for the given $T(nW_m)$ as follows:

- $x = 1$
- $e_k^1 = k + 1$ for $1 \leq k \leq m$
- $e_{k-m}^2 = k + 1$ for $m + 1 \leq k \leq 2m$
- $e_{k-2m}^3 = k + 1$ for $2m + 1 \leq k \leq 3m$
  $\vdots$
- $e_{k-(n-1)m}^n = k + 1$ for $(n-1)m + 1 \leq k \leq nm$
- $g_{1m}^k = k + n(m-1)$ for $1 \leq k \leq n$
- $f_k^1 = k$, $2 \leq k \leq m - 1$
- $f_{k-(m-1)}^2 = k$ for $m \leq k \leq 2(m-1)$
- $f_{k-2(m-1)}^3 = k$ for $2m - 1 \leq k \leq 3(m-1)$
  $\vdots$
- $f_{k-(n-1)(m-1)}^n = k$ for $(m-1)n - m + 2 \leq k \leq n(m-1)$

- $f_1^1 = mn + 1$.
- $a_k^1 = k + 2$ for $1 \leq k \leq m$
- $a_{k-m}^2 = k + 2$ for $m + 1 \leq k \leq 2m$
- $a_{k-2m}^3 = k + 2$ for $2m + 1 \leq k \leq 3m$
  $\vdots$
- $a_{k-(n-1)m}^n = k + 2$ for $(n-1)m + 1 \leq k \leq mn - 1$
- $a_m^n = 2$.

All $d\left(f_i^j\right) = d\left(a_i^j\right)$ but $code\left(f_i^j\right) \neq code\left(a_i^j\right)$, all $a_i^{j'}$s are adjacent to x but all $f_i^{j'}$s are not adjacent to x. Hence, $\chi_{ir}(T(nW_m)) \leq mn + 1$. The set $\left\{x, e_i^j : 1 \leq j \leq n,\right.$ $\left.1 \leq i \leq m\right\}$ produce a clique of order $mn + 1$. Therefore, $\chi_{ir}(T(nW_m)) \geq mn+1$. Thus, $\chi_{ir}(T(nW_m)) = mn+1$. □

Theorem 2.3. *For $m \geq 4$ and $n \geq 1$, the irregular chromatic number of n-wheel graph is $\chi_{ir}(L(nW_m)) = mn$.*

Proof. The vertex set $V\left[L\left(nW_m\right)\right] = \left\{e_i^j, f_i^j, g_{1m}^j :\right.$ $\left.1 \leq j \leq n, 1 \leq i \leq m\right\}$, where e_i^j are the vertices convertion of these edges xa_i^j for $1 \leq j \leq n$, $1 \leq i \leq m$, f_i^j are the vertices convertion of these rim edges $a_i^j a_{i+1}^j$ of nW_m for $1 \leq j \leq n$, $1 \leq i \leq m-1$ and g_{1m}^j are the vertices convertion of these rim edges $a_1^j a_m^j$ of nW_m for $1 \leq j \leq n$. Consider the mn-coloring for the given $L(nW_m)$ as follows:

- $e_k^1 = k$ for $1 \leq k \leq m$
- $e_{k-m}^2 = k$ for $m + 1 \leq k \leq 2m$

- $e^3_{k-2m} = k$ for $2m+1 \le k \le 3m$
  $\vdots$
- $e^n_{k-(n-1)m} = k$ for $(n-1)\,m+1 \le k \le nm$
- $g^k_{1m} = k-1+n(m-1)$ for $1 \le k \le n$
- $f^1_k = k-1,\ 2 \le k \le m-1$
- $f^2_{k-(m-1)} = k-1$ for $m \le k \le 2(m-1)$
- $f^3_{k-2(m-1)} = k-1$ for $2m-1 \le k \le 3(m-1)$
  $\vdots$
- $f^n_{k-(n-1)(m-1)} = k-1$ for $(m-1)\,n-m+2 \le k \le n(m-1)$
- $f^1_1 = mn.$

All $d\left(f^j_i\right) \neq d\left(e^j_i\right)$ which implies that *code* $\left(f^j_i\right) \neq$ *code* $\left(e^j_i\right)$. Hence, $\chi_{ir}(L(nW_m)) \le mn$. The set $\left\{e^j_i : 1 \le j \le n,\right.$ $\left.1 \le i \le m\right\}$ produce a clique of order mn. Therefore, $\chi_{ir}(L(nW_m)) \ge mn$. Thus, $\chi_{ir}(L(nW_m)) = mn$. □

Theorem 2.4. *For $m, n \ge 3$, the irregular chromatic number of n-wheel graph is $\chi_{ir}(C(nW_m)) = n\left\lceil \frac{m}{2} \right\rceil$.*

Proof. The vertex set $V\left[C\left(nW_m\right)\right] = \left\{x, a^j_i, e^j_i, f^j_i, g^j_{1m} :\right.$ $\left.1 \le j \le n,\ 1 \le i \le m\right\}$, where e^j_i are the vertices on the edges xa^j_i for $1 \le j \le n,\ 1 \le i \le m$, f^j_i are the vertices on the rim edges $a^j_i a^j_{i+1}$ of nW_m for $1 \le j \le n,\ 1 \le i \le m-1$ and g^j_{1m} are the vertices on the rim edges $a^j_1 a^j_m$ of nW_m for

$1 \le j \le n$.
The coloring procedure of $C(nW_m)$ as follows:

- $x = 1$

For $m = 3, 4$,

- $g^1_{1m} = n\left\lceil \frac{m}{2} \right\rceil - 1$
- $g^2_{1m} = 1$
- $g^k_{1m} = 2$ for $3 \le k \le n$

For $m \ge 5$,

- $g^k_{1m} = n\left\lceil \frac{m}{2} \right\rceil$ for $1 \le k \le n - 1$
- $g^n_{1m} = 1$

Case 1: m is even

- $f^j_i = n\left(\frac{m}{2}\right)$ for $1 \le j \le n - 1,\ 1 \le i \le m$
- $f^n_i = n\left(\frac{m}{2}\right)$ for $1 \le i \le m - 2$
- $f^n_m = f^n_{m-1} = 1$
- $e^1_{2k-1} = k + 1$ for $1 \le k \le \frac{m}{2}$
- $e^2_{2k-1} = \frac{m}{2} + k + 1$ for $1 \le k \le \frac{m}{2}$
- $e^3_{2k-1} = m + k + 1$ for $1 \le k \le \frac{m}{2}$
  $\vdots$
- $e^n_{2k-1} = (n-1)\frac{m}{2} + k + 1$ for $1 \le k \le \frac{m}{2} - 1$
- $e^n_{m-1} = 2$

- $e^1_{2k} = k + 2$ for $1 \leq k \leq \frac{m}{2}$
- $e^2_{2k} = \frac{m}{2} + k + 2$ for $1 \leq k \leq \frac{m}{2}$
- $e^3_{2k} = m + k + 2$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $e^n_{2k} = (n-1)\frac{m}{2} + k + 2$ for $1 \leq k \leq \frac{m}{2} - 2$
- $e^n_{m-2} = 2$
- $e^n_m = 3$
- $a^1_{2k-1} = k$ for $1 \leq k \leq \frac{m}{2}$
- $a^2_{2k-1} = \frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^3_{2k-1} = m + k$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $a^n_{2k-1} = (n-1)\frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^1_{2k} = k$ for $1 \leq k \leq \frac{m}{2}$
- $a^2_{2k} = \frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^3_{2k} = m + k$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $a^n_{2k} = (n-1)\frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$

Case 2: m is odd

- $f^j_i = n\left\lceil\frac{m}{2}\right\rceil$ for $1 \leq j \leq n-1,\ 1 \leq i \leq m$
- $f^n_i = n\left\lceil\frac{m}{2}\right\rceil$ for $1 \leq i \leq m-1$

- $f_m^n = 1$
- $e_{2k-1}^1 = k + 1$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$
- $e_{2k-1}^2 = \lceil \frac{m}{2} \rceil + k + 1$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$
- $e_{2k-1}^3 = 2 \lceil \frac{m}{2} \rceil + k + 1$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$

  $\vdots$
- $e_{2k-1}^n = (n-1) \lceil \frac{m}{2} \rceil + k + 1$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil - 1$
- $e_m^n = 2$
- $e_{2k}^1 = k + 2$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$
- $e_{2k}^2 = \lceil \frac{m}{2} \rceil + k + 2$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$
- $e_{2k}^3 = 2 \lceil \frac{m}{2} \rceil + k + 2$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$

  $\vdots$
- $e_{2k}^n = (n-1) \lceil \frac{m}{2} \rceil + k + 2$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor - 1$
- $e_{m-1}^n = 2$
- $a_{2k-1}^1 = k$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$
- $a_{2k-1}^2 = \lceil \frac{m}{2} \rceil + k$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$
- $a_{2k-1}^3 = 2 \lceil \frac{m}{2} \rceil + k$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$

  $\vdots$
- $a_{2k-1}^n = (n-1) \lceil \frac{m}{2} \rceil + k$ for $1 \leq k \leq \lceil \frac{m}{2} \rceil$
- $a_{2k}^1 = k$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$
- $a_{2k}^2 = \lceil \frac{m}{2} \rceil + k$ for $1 \leq k \leq \lfloor \frac{m}{2} \rfloor$

- $a^3_{2k} = 2\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \le k \le \left\lfloor \frac{m}{2} \right\rfloor$

  $\vdots$

- $a^n_{2k} = (n-1)\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \le k \le \left\lfloor \frac{m}{2} \right\rfloor$

All $d\left(a^j_i\right) \neq d\left(e^j_i\right)$ which implies that $code\left(a^j_i\right) \neq code\left(e^j_i\right)$. It is clear that the above coloring is an irregular coloring. Hence, $\chi_{ir}(C(nW_m)) \le n\left\lceil \frac{m}{2} \right\rceil$. The set $\left\{a^j_{2i-1} : 1 \le j \le n,\right.$ $\left.1 \le i \le \left\lceil \frac{m}{2} \right\rceil\right\}$ produce a clique of order $n\left\lceil \frac{m}{2} \right\rceil$, thus $\chi_{ir}(C(nW_m)) \ge n\left\lceil \frac{m}{2} \right\rceil$. Therefore, $\chi_{ir}(C(nW_m)) = n\left\lceil \frac{m}{2} \right\rceil$. □

3 Irregular Coloring of middle graph, total graph, central graph and line graph of n-fan Graph

Theorem 3.1. *For $m \ge 4$ and $n \ge 1$, the irregular chromatic number of n-fan graph is $\chi_{ir}(M(nF_m)) = mn + 1$.*

Proof. The vertex set $V\left[M\left(nF_m\right)\right] = \left\{x, a^j_i, e^j_i, f^j_i :\right.$ $\left.1 \le j \le n,\ 1 \le i \le m\right\}$, where e^j_i are the vertices on the edges xa^j_i for $1 \le j \le n,\ 1 \le i \le m$ and f^j_i are the vertices on the rim edges $a^j_i a^j_{i+1}$ of nF_m for $1 \le j \le n,\ 1 \le i \le m-1$.
Consider the $(mn+1)$-coloring for the given $M(nF_m)$ as follows:

- $x = 1$
- $e^1_k = k + 1$ for $1 \le k \le m$
- $e^2_{k-m} = k + 1$ for $m + 1 \le k \le 2m$

- $e^3_{k-2m} = k+1$ for $2m+1 \leq k \leq 3m$

  $\vdots$

- $e^n_{k-(n-1)m} = k+1$ for $(n-1)\,m+1 \leq k \leq nm$
- For $1 \leq i \leq m$, $1 \leq j \leq n$ $a^j_i = 1$
- $f^1_k = k$, $2 \leq k \leq m-1$
- $f^2_{k-(m-1)} = k$ for $m \leq k \leq 2(m-1)$
- $f^3_{k-2(m-1)} = k$ for $2m-1 \leq k \leq 3(m-1)$

  $\vdots$

- $f^n_{k-(n-1)(m-1)} = k$ for $(m-1)\,n - m + 2 \leq k \leq n(m-1)$
- $f^1_1 = mn+1$.

All $d\left(f^j_i\right) \neq d\left(e^j_i\right)$ which implies that $code\left(f^j_i\right) \neq code\left(e^j_i\right)$. Hence, $\chi_{ir}(M(nF_m)) \leq mn+1$. The set $\left\{x, e^j_i : 1 \leq j \leq n, 1 \leq i \leq m\right\}$ produce a clique of order $mn+1$. Therefore, $\chi_{ir}(M(nF_m)) \geq mn+1$. Thus, $\chi_{ir}(M(nF_m)) = mn+1$. □

Theorem 3.2. *For $m \geq 4$ and $n \geq 1$, the irregular chromatic number of n-fan graph is $\chi_{ir}(T(nF_m)) = mn+1$.*

Proof. The vertex set $V\left[T\left(nF_m\right)\right] = \left\{x, a^j_i, e^j_i, f^j_i : 1 \leq j \leq n,\ 1 \leq i \leq m\right\}$, where e^j_i are the vertices on the edges xa^j_i for $1 \leq j \leq n$, $1 \leq i \leq m$ and f^j_i are the vertices on the rim edges $a^j_i a^j_{i+1}$ of nF_m for $1 \leq j \leq n$, $1 \leq i \leq$

$m-1$.
Consider the $(mn+1)$-coloring for the given $T(nF_m)$ as follows:

- $x = 1$
- $e^1_k = k+1$ for $1 \le k \le m$
- $e^2_{k-m} = k+1$ for $m+1 \le k \le 2m$
- $e^3_{k-2m} = k+1$ for $2m+1 \le k \le 3m$
  $\vdots$
- $e^n_{k-(n-1)m} = k+1$ for $(n-1)\,m+1 \le k \le nm$
- $f^1_k = k$, $2 \le k \le m-1$
- $f^2_{k-(m-1)} = k$ for $m \le k \le 2(m-1)$
- $f^3_{k-2(m-1)} = k$ for $2m-1 \le k \le 3(m-1)$
  $\vdots$
- $f^n_{k-(n-1)(m-1)} = k$ for $(m-1)\,n-m+2 \le k \le n(m-1)$
- $f^1_1 = mn+1$.
- $a^1_k = k+2$ for $1 \le k \le m$
- $a^2_{k-m} = k+2$ for $m+1 \le k \le 2m$
- $a^3_{k-2m} = k+2$ for $2m+1 \le k \le 3m$
  $\vdots$
- $a^n_{k-(n-1)m} = k+2$ for $(n-1)m+1 \le k \le mn-1$

- $a_m^n = 2$.

All $d\left(f_i^j\right) = d\left(a_i^j\right)$ but *code* $\left(f_i^j\right) \neq$ *code* $\left(a_i^j\right)$, all $a_i^{j\prime}$s are adjacent to x but all $f_i^{j\prime}$s are not adjacent to x. Hence, $\chi_{ir}(T(nF_m)) \leq mn + 1$. The set $\left\{x, e_i^j : 1 \leq j \leq n,\right.$ $\left.1 \leq i \leq m\right\}$ produce a clique of order $mn + 1$. Therefore, $\chi_{ir}(T(nF_m)) \geq mn+1$. Thus, $\chi_{ir}(T(nF_m)) = mn+1$. □

Theorem 3.3. *For $m \geq 4$ and $n \geq 1$, the irregular chromatic number of n-fan graph is $\chi_{ir}(L(nF_m)) = mn$.*

Proof. The vertex set $V\left[L\left(nF_m\right)\right] = \left\{e_i^j, f_i^j : 1 \leq j \leq n,\right.$ $\left.1 \leq i \leq m\right\}$, where e_i^j are the vertices convertion of the edges xa_i^j for $1 \leq j \leq n$, $1 \leq i \leq m$ and f_i^j are the vertices convertion of the rim edges $a_i^j a_{i+1}^j$ of nF_m for $1 \leq j \leq n$, $1 \leq i \leq m-1$.
Consider the mn-coloring for the given $L(nF_m)$ as follows:

- $e_k^1 = k$ for $1 \leq k \leq m$
- $e_{k-m}^2 = k$ for $m+1 \leq k \leq 2m$
- $e_{k-2m}^3 = k$ for $2m+1 \leq k \leq 3m$

  ⋮
- $e_{k-(n-1)m}^n = k$ for $(n-1)\,m + 1 \leq k \leq nm$
- $g_{1m}^k = k - 1 + n(m-1)$ for $1 \leq k \leq n$
- $f_k^1 = k - 1$, $2 \leq k \leq m-1$
- $f_{k-(m-1)}^2 = k - 1$ for $m \leq k \leq 2(m-1)$

- $f^3_{k-2(m-1)} = k-1$ for $2m-1 \leq k \leq 3(m-1)$
  $\vdots$
- $f^n_{k-(n-1)(m-1)} = k-1$ for $(m-1)\,n - m + 2 \leq k \leq n(m-1)$
- $f^1_1 = mn$.

All $d\left(f^j_i\right) \neq d\left(e^j_i\right)$ which implies that $code\left(f^j_i\right) \neq code\left(e^j_i\right)$. Hence, $\chi_{ir}(L(nF_m)) \leq mn$. The set $\left\{e^j_i : 1 \leq j \leq n,\right.$ $\left.1 \leq i \leq m\right\}$ produce a clique of order mn. Therefore, $\chi_{ir}(L(nF_m)) \geq mn$. Thus, $\chi_{ir}(L(nF_m)) = mn$. □

Theorem 3.4. *For $m, n \geq 3$, the irregular chromatic number of n-fan graph is $\chi_{ir}(C(nF_m)) = n\left\lceil \frac{m}{2} \right\rceil$.*

Proof. The vertex set $V\left[C\left(nF_m\right)\right] = \left\{x, a^j_i, e^j_i, f^j_i :\right.$ $\left.1 \leq j \leq n,\ 1 \leq i \leq m\right\}$, where e^j_i are the vertices converted from the edges xa^j_i for $1 \leq j \leq n,\ 1 \leq i \leq m$ and f^j_i are the vertices converted from the rim edges $a^j_i a^j_{i+1}$ of nF_m for $1 \leq j \leq n,\ 1 \leq i \leq m-1$.
The coloring procedure of $C(nF_m)$ as follows:

- $x = 1$

Case 1: m is even

- $f^j_i = n\left(\frac{m}{2}\right)$ for $1 \leq j \leq n-1,\ 1 \leq i \leq m$
- $f^n_i = n\left(\frac{m}{2}\right)$ for $1 \leq i \leq m-2$
- $f^n_m = f^n_{m-1} = 1$
- $e^1_{2k-1} = k+1$ for $1 \leq k \leq \frac{m}{2}$

- $e^2_{2k-1} = \frac{m}{2} + k + 1$ for $1 \leq k \leq \frac{m}{2}$
- $e^3_{2k-1} = m + k + 1$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $e^n_{2k-1} = (n-1)\frac{m}{2} + k + 1$ for $1 \leq k \leq \frac{m}{2} - 1$
- $e^n_{m-1} = 2$
- $e^1_{2k} = k + 2$ for $1 \leq k \leq \frac{m}{2}$
- $e^2_{2k} = \frac{m}{2} + k + 2$ for $1 \leq k \leq \frac{m}{2}$
- $e^3_{2k} = m + k + 2$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $e^n_{2k} = (n-1)\frac{m}{2} + k + 2$ for $1 \leq k \leq \frac{m}{2} - 2$
- $e^n_{m-2} = 2$
- $e^n_{m} = 3$
- $a^1_{2k-1} = k$ for $1 \leq k \leq \frac{m}{2}$
- $a^2_{2k-1} = \frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^3_{2k-1} = m + k$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$
- $a^n_{2k-1} = (n-1)\frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^1_{2k} = k$ for $1 \leq k \leq \frac{m}{2}$
- $a^2_{2k} = \frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$
- $a^3_{2k} = m + k$ for $1 \leq k \leq \frac{m}{2}$

  $\vdots$

- $a_{2k}^{n} = (n-1)\frac{m}{2} + k$ for $1 \leq k \leq \frac{m}{2}$

Case 2: m is odd

- $f_i^j = n\left\lceil \frac{m}{2} \right\rceil$ for $1 \leq j \leq n-1,\ 1 \leq i \leq m$
- $f_i^n = n\left\lceil \frac{m}{2} \right\rceil$ for $1 \leq i \leq m-1$
- $f_m^n = 1$
- $e_{2k-1}^{1} = k+1$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
- $e_{2k-1}^{2} = \left\lceil \frac{m}{2} \right\rceil + k + 1$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
- $e_{2k-1}^{3} = 2\left\lceil \frac{m}{2} \right\rceil + k + 1$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
  $\vdots$
- $e_{2k-1}^{n} = (n-1)\left\lceil \frac{m}{2} \right\rceil + k + 1$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil - 1$
- $e_m^n = 2$
- $e_{2k}^{1} = k+2$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
- $e_{2k}^{2} = \left\lceil \frac{m}{2} \right\rceil + k + 2$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
- $e_{2k}^{3} = 2\left\lceil \frac{m}{2} \right\rceil + k + 2$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
  $\vdots$
- $e_{2k}^{n} = (n-1)\left\lceil \frac{m}{2} \right\rceil + k + 2$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor - 1$
- $e_{m-1}^{n} = 2$
- $a_{2k-1}^{1} = k$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
- $a_{2k-1}^{2} = \left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$

- $a^3_{2k-1} = 2\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
  $\vdots$
- $a^n_{2k-1} = (n-1)\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lceil \frac{m}{2} \right\rceil$
- $a^1_{2k} = k$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
- $a^2_{2k} = \left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
- $a^3_{2k} = 2\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$
  $\vdots$
- $a^n_{2k} = (n-1)\left\lceil \frac{m}{2} \right\rceil + k$ for $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$

All $d\left(a^j_i\right) \neq d\left(e^j_i\right)$ which implies that $code\left(a^j_i\right) \neq code\left(e^j_i\right)$. It is clear that the above coloring is an irregular coloring. Hence, $\chi_{ir}(C(nF_m)) \leq n\left\lceil \frac{m}{2} \right\rceil$. The set $\left\{a^j_{2i-1} : 1 \leq j \leq n,\ 1 \leq i \leq \left\lceil \frac{m}{2} \right\rceil\right\}$ produce a clique of order $n\left\lceil \frac{m}{2} \right\rceil$, thus $\chi_{ir}(C(nF_m)) \geq n\left\lceil \frac{m}{2} \right\rceil$. Therefore, $\chi_{ir}(C(nF_m)) = n\left\lceil \frac{m}{2} \right\rceil$. □

4 Irregular Coloring of Splitting graph and Mycielskian graph of any graph

Theorem 4.1. *For any graph G, the irregular chromatic number $\chi_{ir}(S(G)) = \chi_{ir}(G)$.*

Proof. Let $\{v_i : (1 \leq i \leq n)\}$ be the vertex set of the graph G and assume that graph G has an irregular coloring partition. Let $\left\{v_i, v_i^{'} : (1 \leq i \leq n)\right\}$ be the vertex set of splitting graph of G, i.e., $S(G)$. The degree of the vertices $d(v_i) = 2d(v_i^{'})$, assume that same colors assigned to v_i and

$v_i^{'}$. Here, $d(v_i) \neq d(v_i^{'})$. By equation (2), which implies that $code(v_i) \neq code(v_i^{'})$. Hence the irregular chromatic number, $\chi_{ir}(S(G)) = \chi_{ir}(G)$. □

Theorem 4.2. *For any graph G, the irregular chromatic number $\chi_{ir}(\mu(G)) = \chi_{ir}(G) + 1$.*

Proof. Let $\{v_i : (1 \leq i \leq n)\}$ be the vertex set of the graph G and assume that graph G has an irregular coloring partition. Let $\left\{v_i, v_i^{'}, w : (1 \leq i \leq n)\right\}$ be the vertex set of mycielskian graph of G, i.e., $\mu(G)$. The degree of the vertices $d(v_i) = 2(d(v_i^{'})-1)$, assume that same colors to v_i and $v_i^{'}$ and a new color to the vertex w. Here, $d(v_i) \neq d(v_i^{'})$. By equation (2), which implies that $code(v_i) \neq code(v_i^{'})$. Hence the irregular chromatic number,
$\chi_{ir}(\mu(G)) = \chi_{ir}(G) + 1$.

□

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